Hopf bifurcation in a leaky faucet experiment

R. D. Pinto, W. M. Gonçalves, J. C. Sartorelli, and M. J. de Oliveira
Instituto de Física, Universidade de São Paulo, Caixa Postal 66 318, CEP 05389-970, São Paulo, Brazil
(Received 9 March 1995)

In a leaky faucet experiment an inverse Hopf bifurcation was observed, as one increases the water flux, before the occurrence of the continuous flow. For values of the drop rate smaller than the critical drop rate, the movement is periodic or quasiperiodic with a finite amplitude of the time series. At the critical point the amplitude of the time series vanishes and it suggests the bifurcation point as the threshold of the continuous flow.

PACS number(s): 05.45.+b, 47.52.+j

In a leaky faucet experiment [1–6], complex dynamical behavior has been observed which includes quasiperiodicity, intermittencies, and chaos as well as boundary crisis. Recently [6], the occurrence of a boundary crisis has been detected experimentally in drop-to-drop interval time series where a sudden change occurs from a chaotic movement to a period-5 movement. Starting from this late state ($f \approx 38.42$ drops/s) and increasing the water flow rate we got a sequence of 17 time series corresponding to different drop rates. The sequence finishes when the water flux at the laser level becomes continuous ($f \approx 39.7$ drops/s). The evolution from periodic to quasiperiodic movement is shown in Fig. 1 where return maps t_{n+1} vs t_n of nine among the 17 drop rates are displayed. The quasiperiodic character can be inferred during the construction of the return maps, the points appearing in a clockwise sense. In Fig. 2 are shown the Fourier transforms of these 17 pseudotemporal series.

In Fig. 3 we show the periods of the pseudotemporal series, determined from the Fourier analysis, as a function of the mean drop rate. The first six series have a period of $\tau_0 = 5$ drops and the quasiperiodic sequence starts around the seventh series. The period τ increases with the drop rate f toward the value $\tau_1 = 6$ drops. Since the period of each series might be an irrational number we will make use of a Farey tree construction [7] to find the best rational number which describes the period. We organize the rationals between $\tau_0 = 5$ and $\tau_1 = 6$ drops by constructing the fractions $\tau(p,q) = (p\tau_0 + q\tau_1)/(p+q)$ where p and q are integers. They are $\tau(1,0) = 5$, $\tau(0,1) = 6$, $\tau(1,1) = 11/2$, $\tau(2,1) = 16/3$, $\tau(1,2) = 17/3$ drops, and so on. To find the components (p,q) for a given time series t_n, $n = 1,2,3,...,N$, we proceed as follows. We divide this series into a certain number of

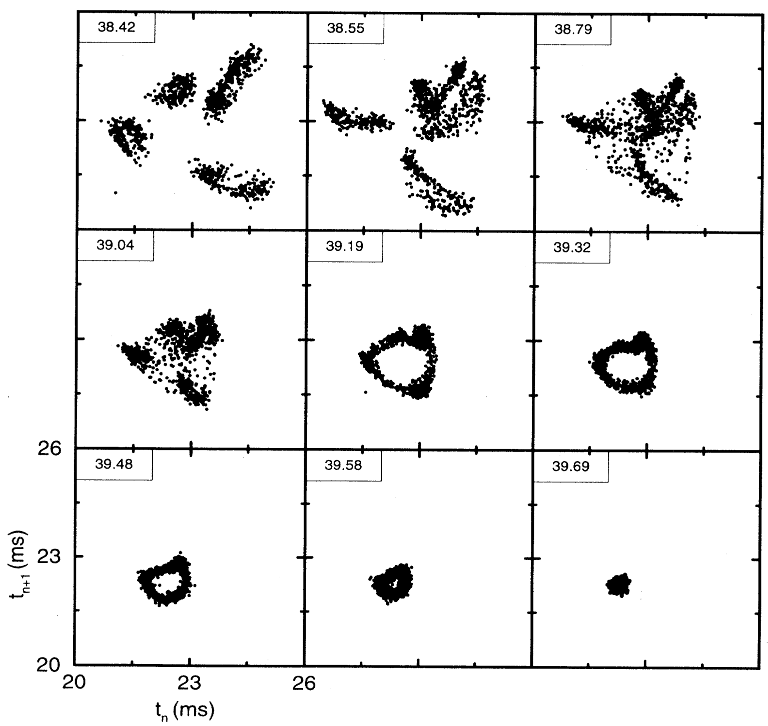

FIG. 1. Nine-returns maps t_{n+1} vs t_n showing the evolution from periodic behavior to quasiperiodic movement. The insets at the top left are the mean drop rate value. Times in ms.

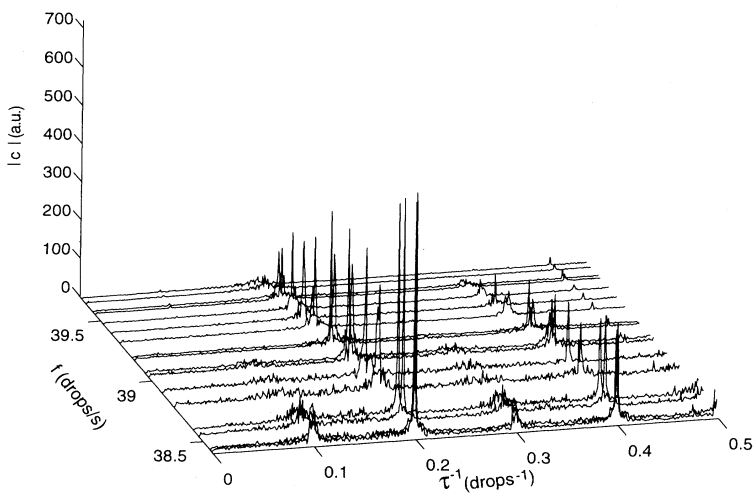

FIG. 2. The Fourier transform amplitudes of the 17 pseudotemporal series as a function of the mean drop rate.

subseries of length T. We superpose all the subseries by shifting them to the same interval $[1,T]$ and by adding them up to get a new series of length T.

The new time series $\bar{t}_\ell(T)$ is defined by

$$\bar{t}_\ell(T) = \frac{1}{K+1}\sum_{k=0}^{K} t_{kT+\ell}, \tag{1}$$

where K equals the integer part of $(N/T - 1)$ and N is the total number of drops in the whole time series. From this new time series we calculate the average

$$m(T) = \frac{1}{T}\sum_{\ell=1}^{T}\bar{t}_\ell(T) \tag{2}$$

and the mean square deviation $d(T)$ defined by

$$d(T) = \sqrt{\frac{1}{T}\sum_{\ell=1}^{T}[\bar{t}_\ell(T) - m(T)]^2}. \tag{3}$$

Due to the superposition of the subseries, the function $d(T)$ displays a multitude of peaks which might be of the same height, as shown in Fig. 4, or of different heights, as can be seen in Fig. 5. Each peak can be associated with a unique value of the pair of Farey components (p,q); if there is more than one possible pair, we choose the one with $p+q=\min$, but $p\cdot q\neq 0$ for quasiperiodic series. The best pair is chosen as the one corresponding to the highest local peak near the period given by the Fourier analysis. If there is more than one peak with the same height, we choose the one corresponding to the least sum $p+q$.

For periodic series, the function $d(T)$ shows peaks of the same intensity when T is a multiple of the fundamental period. This is shown in Fig. 4 for the case where

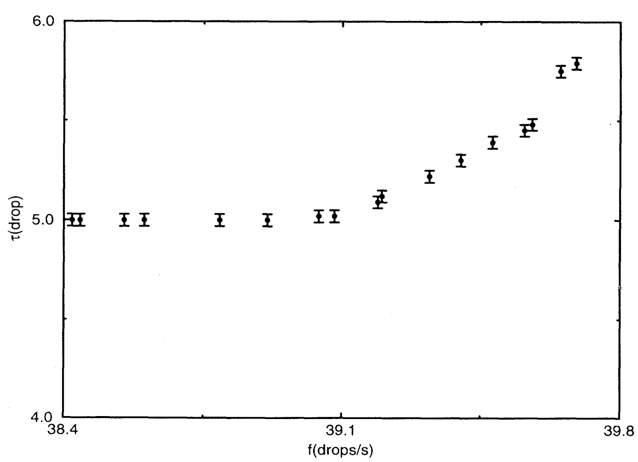

FIG. 3. Period of the 17 pseudotemporal series obtained from the Fourier transforms versus the mean drop rate.

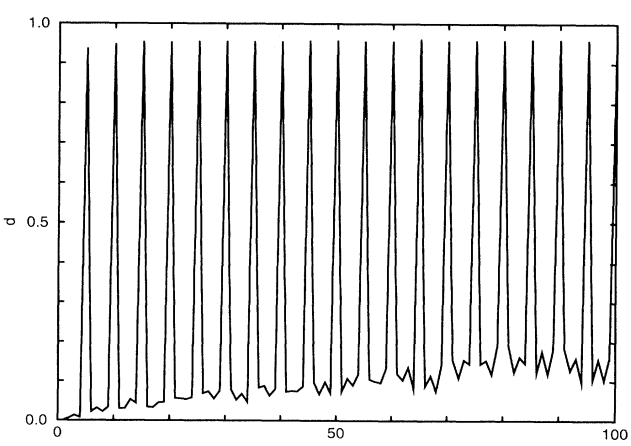

FIG. 4. d vs T of a periodic series with $\tau = 5$ drops.

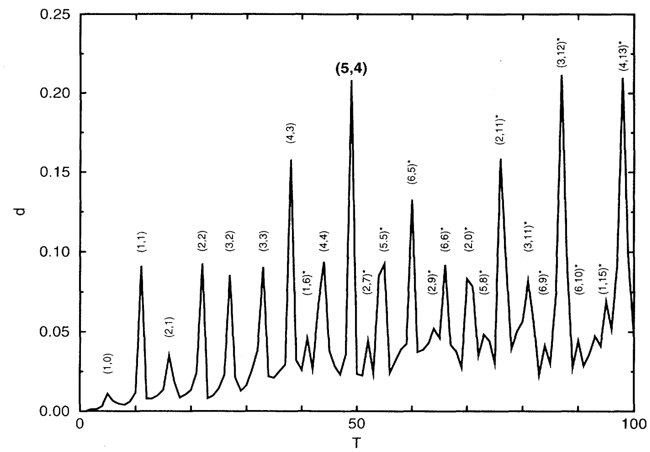

FIG. 5. d vs T of a quasiperiodic series, and the Farey components (p,q) (* means that there is more than one possible pair). The first maximum occurs at the pair in boldface, at $\tau = 49/9$ drops.

$f = 38.55$ drops/s. The values of (p,q) for the peaks are $(1,0)$, $(2,0)$, $(3,0)$, etc. We choose the first one which corresponds to $\tau_0 = 5$ drops. For a quasiperiodic series, as shown in Fig. 5 for the case of $f = 39.56$ drops/s, the majority of the peaks are of different intensity. The three largest peaks are of the same height and are associated with the following Farey components: $(5,4)$, $(3,12)$, $(4,13)$. We choose the first one which gives $\tau = 49/9$. Table I shows the estimation of τ given by this procedure with the corresponding Farey components (p,q) together with τ determined from the Fourier analysis.

A Hopf bifurcation [8] is described by the two-dimensional map $(r,\theta) \to (r',\theta')$ where

$$r' = [1 + d(\mu - \mu_0)]r + ar^3 \tag{4}$$

and

$$\theta' = \theta + c + br^2, \tag{5}$$

where μ is the control parameter, μ_0 is the critical control parameter, and a, b, c, and d are constants. In the present case it is appropriate to choose d to be negative and a to be positive. Thus, as long as $\mu < \mu_0$ the map displays a limit cycle of radius r_0 given by

$$r_0^2 = \frac{d}{a}(\mu - \mu_0) \tag{6}$$

and the rotation number

$$\omega = c + br_0^2 \tag{7}$$

or yet

$$\omega = c + \frac{b|d|}{a}(\mu - \mu_0). \tag{8}$$

The corresponding period τ will be $\tau = 2\pi/\omega$.

We have identified the control parameter as the drop rate and the radius of the limit cycle as half the mean amplitude of the pseudotime series. More precisely we have set $\mu = f$ and $r_0 = (t_{max} - t_{min})/2$, where t_{max} (t_{min}) is the mean value of t calculated in the last (first) bin of a ten-bin histogram. In Fig. 6 we show the plot of r_0^2 versus f. From a linear fitting, predicted by Eq. (6), we obtained the critical drop rate $f_0 = 39.705$ drops/s. Figure 7 shows the rotation number $1/\tau$ versus r_0^2 together with a linear fitting predicted by Eq. (8). The data in both figures are well aligned, indicating that a Hopf bifurcation indeed takes place. The bifurcation is actually

TABLE I. The second column shows τ as obtained from the Fourier analysis as a function of the drop rate. The last column displays $\tau_a = (5p + 6q)/(p+q)$ where the pairs (p,q) are the Farey components obtained from the $d(T)$ analysis.

f (drops/s)	τ (drops)$\pm$0.03	p	q	τ_a
38.42	5.00	1	0	5
38.44	5.00	1	0	5
38.55	5.00	1	0	5
38.60	5.00	1	0	5
38.79	5.00	1	0	5
38.91	5.00	1	0	5
39.04	5.02	1	0	5
39.08	5.02	1	0	5
39.19	5.09	9	1	5.1
39.20	5.12	7	1	5.125
39.32	5.22	7	2	5.222...
39.40	5.30	5	2	5.285...
39.48	5.39	3	2	5.4
39.56	5.45	5	4	5.444...
39.58	5.48	1	1	5.5
39.65	5.75	1	3	5.75
39.69	5.79	1	4	5.8

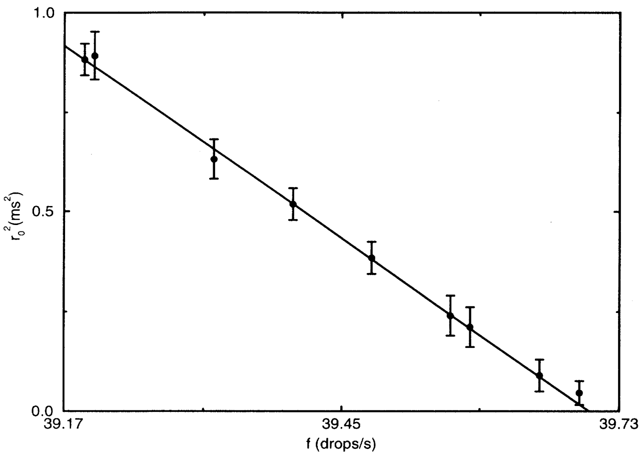

FIG. 6. r_0^2 vs f. The continuous line is the linear curve fit to the experimental data. The critical drop rate is $f_0 = 39.705$ drops/s.

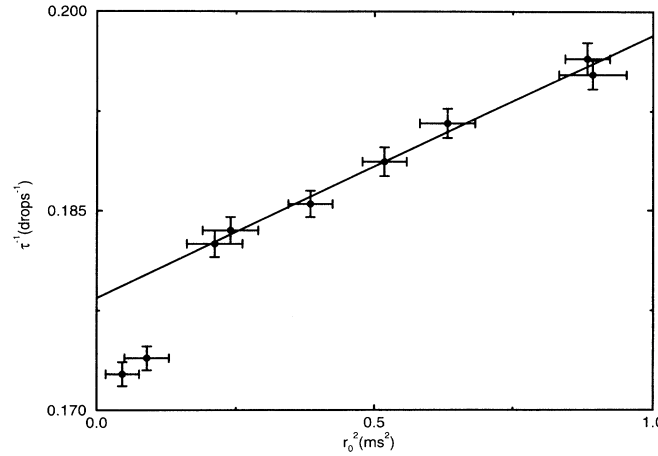

FIG. 7. τ^{-1} vs r_0^2. The two misaligned points are the ones with the least power spectra intensities shown in Fig. 2.

an inverse one since one should *decrease* the drop rate (from a value larger than f_0) to observe the emergence of the limit cycle in the reconstructed Poincaré section.

As our data are discrete, as in a map, the evolution of just one frequency, and its harmonics in the Fourier analysis, is an indication that the bifurcation might be a secondary Hopf (or Neimark) bifurcation [9]. The two points misaligned in Fig. 7, have time series with small signal-to-noise ratio, which makes it difficult to define $1/\tau$ (see relative intensities in Fig. 2). We are improving the experimental apparatus to get a better signal-to-noise ratio.

We have shown that in a leaky faucet experiment an inverse secondary Hopf (or Neimark) bifurcation was observed, as one increases the water flux, before the occurrence of the continuous flow. For values of the drop rate smaller than the critical drop rate, the movement is periodic or quasiperiodic with a finite amplitude of the time series. At the critical point the amplitude of the time series vanishes and we identify the bifurcation point as the threshold of the continuous flow.

This work was partially financed by the Brazilian agencies Finep, CNPq, and Fapesp.

[1] P. Martien, S. C. Pope, P. L. Scott, and R. S. Shaw, Phys. Lett. **110A**, 339 (1985).
[2] H. N. N. Yépez, A. L. S. Brito, C. A. Vargas, and L. A. Vicente, Eur. J. Phys. **10**, 99 (1989).
[3] R. F. Cahalan, H. Leidecher, and G. D. Cahalan, Comput. Phys. **4**, 368 (1990).
[4] X. Wu and Z. A. Schelly, Physica D **40**, 433 (1989).
[5] K. Dreyer and F. R. Hickey, Am. J. Phys. **59**, 619 (1991).
[6] J. C. Sartorelli, W. M. Gonçalves, and R. D. Pinto, Phys. Rev. E **49**, 3963 (1994).
[7] S. Kim and S. Ostlund, Phys. Rev. A **34**, 3426 (1986).
[8] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1986).
[9] J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos, Geometrical Methods for Engineers and Scientists* (John Wiley & Sons, New York, 1986).